A Friendly Amendment of the Theil Index

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Theil's Index

Theil's Index, a version of Shannon's Entropy, is introduced in econometrics as a measure for inequality. It is improperly specified for statistical use. We explore an adjustment of the Theil index by considering...

- Shannon's original Axiomatization
- Theil's (Mis)-Specification via Shannon
- Adjustment and Re-Specification

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Outline

Shannon's Axioms

Shannon represents an 'Information Source' - a random process on a discrete space - as a Markov process.
He supposes a "measure" *H* should have these qualities:

- *H* should be continuous in p_i
- For $p_i = p, \; \forall \; i$, H should monotonically increase
- "If a choice be broken down, the original [*H*] should be the weighted sum"

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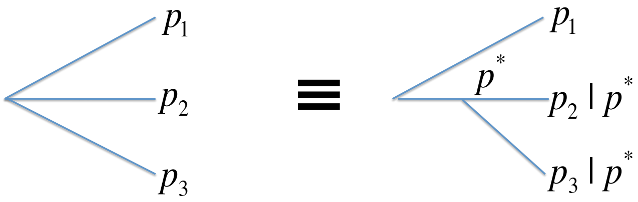
Consistency over conditioning...

$$H(p_1, p_2, p_3) \equiv H(p_1, p^*) + p^* H(p_2|p^*, p_3|p^*)$$



Shannon's Theorem

The only H satisfying the three axioms is of the form (1949)

$$H = -K \sum_{i=1}^{n} p_i \, log \, p_i$$

Other 'Entropies'

- Gibbs Entropy: $S = -k_B \sum p_i \, log \, p_i$; (1872, 1878)
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Theil's Version

A version of Theil's index is

$$T = n^{-1} \sum \frac{x_i}{n^{-1} \sum_j x_j} \, log \, \frac{x_i}{n^{-1} \sum_j x_j}$$

The probability of a particular event/realization is replaced with the income share for a particular element.

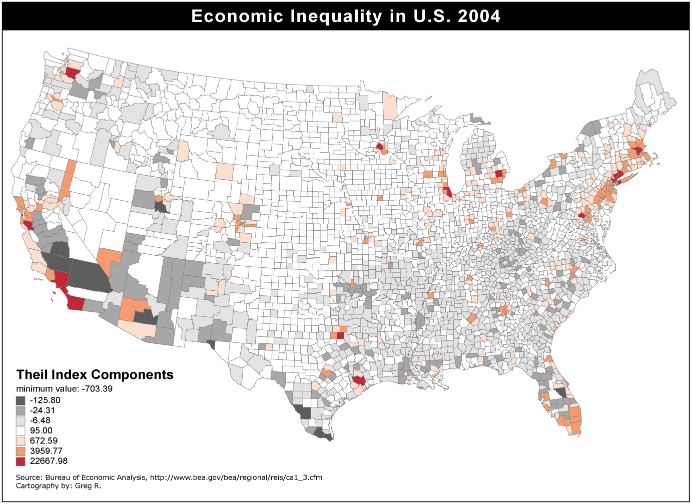
Theil's Decomposition

When collection of elements can be divided into m groups, $g_1, ..., g_m$ each with n_j elements ($=$ individuals); $G = \bigcup g_j$, $g_j \cap g_j^* = \emptyset$, $\forall j \neq j^*$, $\sum_j n_j = n$.

$$T = \sum_G \frac{n_j}{n} \frac{n_j^{-1} \sum_{g_j} x_j}{n^{-1} \sum_G x_j} log \frac{n_j^{-1} \sum_{g_j} x_j}{n^{-1} \sum_G x_j} +$$

$$\sum_G \frac{n_j^{-1} \sum_{g_j} x_j}{n^{-1} \sum_G x_j} \cdot n_j^{-1} \sum_{g_j} \frac{x_j}{n^{-1} \sum_G x_j} log \frac{x_j}{n^{-1} \sum_G x_j}$$

Illustration



Comments

- The probability of a particular event/realization is replaced with the income share for a particular individual.
- We were measuring prob mass of events, now we are measuring the 'size' of individual
- In this sense: The individual is the event, and the income share is the prob mass

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- The lack of a true event space (σ-algebra) yields a degenerate probability model...
- Though heuristic is consistent: n-dimensional simplex / Liouville family of distributions.
- But this belies a "deeper confusion" about Entropy and Probability [Jaynes (1965), *American Journal of Physics*].

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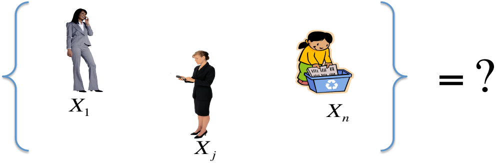
Illustration

$$\Omega = \left\{ E_1 \quad E_j \quad E_n \right\}$$



Shannon's Axiomatization

Notice:

$$T = log(n) - H$$

and that Shannon's original proof specified

$$H = -K \sum_{i=1}^{n} p_i \; log \; p_i$$

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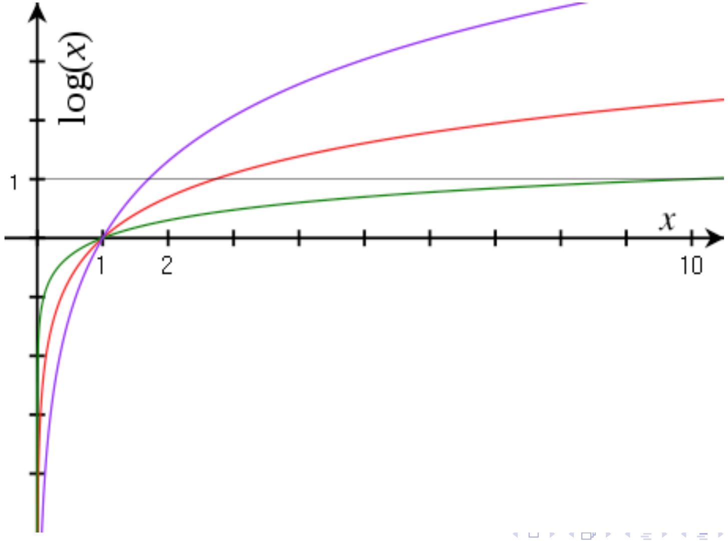
Skew-Sensitivity

One can arbitrarily choose partitions that guarantee across group sum dominates within groups sum.
Merely choose a collection G^* such that $x_j > n_j^{-1} \sum_{g_j} x_j$ for groups $g_1, \ldots, g_{[n/m]}$.
On this collection

$$A \equiv log \frac{n_j^{-1} \sum_{g_j} x_j}{n^{-1} \sum_G x_j} < log \frac{x_j}{n^{-1} \sum_G x_j} \equiv B$$

obscuring within group effects.

Illustration



Log base characterizes 'Granularity'

As well, there exist collections G^* for every choice of b such that on G^* / g^*

$$A, B > 1$$

but

$$A, B < 1$$

on g^*

The Amnesia of (Probability) Measure

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Adjust b Rocke and Durbin Log-Linear hybridization; Box-Cox, Tukey families of transforms.

Rank Transform Estimate empirical CDF across and within groups and substitute for raw income shares

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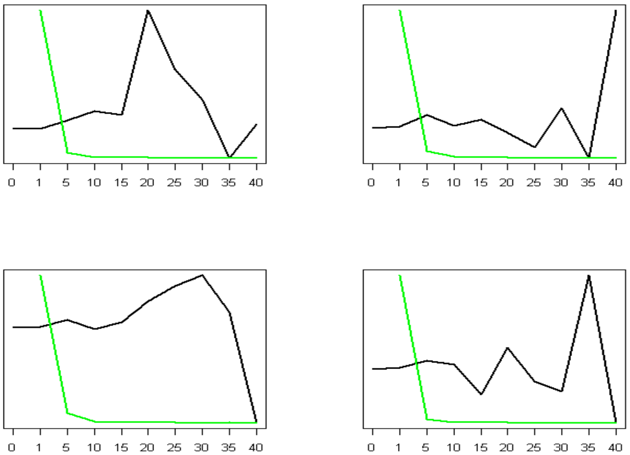
Groups of Equal Mean Uniform Discrete Random Variables



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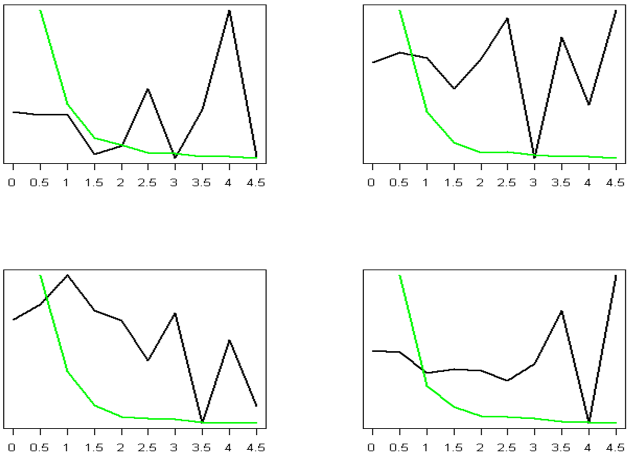

Groups of Equal Mean Uniform Discrete Random Variables, *b* is min.