A Friendly Amendment of the Theil Index

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Theil's Index, a version of Shannon's Entropy, is introduced in econometrics as a measure for inequality. It is improperly specified for statistical use. We explore an adjustment of the Theil index by considering...

- Shannon's original Version
- Theil's (Mis)-Specification via Shannon
- A Friendly Amendment...
- A Test for Equi-Inequality?

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Outline

Shannon's Axioms

Shannon represents an 'Information Source' - a random process on a discrete space - as a Markov process.
He supposes a "measure" H should have these qualities:

1. H should be continuous in event probability, p_i
2. For $p_i = p$, $\forall\ i$, H should monotonically increase
3. "If a choice be broken down, the original [H] should be the weighted sum"

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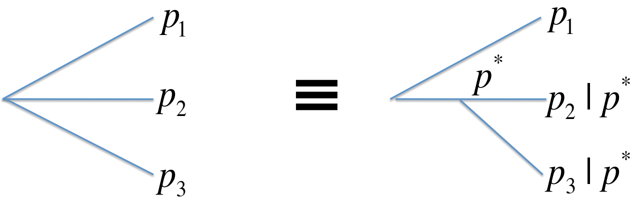
"If a choice be broken down"

Consistency over conditioning...

$$H(p_1, p_2, p_3) \equiv H(p_1, p^*) + p^* H(p_2|p^*, p_3|p^*)$$



Shannon's Theorem

The only H satisfying the three axioms is of the form (1949)

$$H = -K \sum_{i=1}^{n} p_i \, log \, p_i$$

Other 'Entropies'

- Gibbs Entropy: $S = -k_B \sum p_i \, log \, p_i$; (1872, 1878)
- von Neumann Entropy: $S - k_B Tr[\rho \, log_e \, \rho]$

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Entropy's Career

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Theil's Version

A version of Theil's index is

$$T = n^{-1} \sum_{i=1}^{n} r_i \, log \, r_i \tag{1}$$

The probability of a particular event/realization is replaced with the income share for a particular element.

$$r_i = \frac{x_i}{\overline{x}} \tag{2}$$

Theil's Version

Theil's construction suggests

$$H = -\sum_{i=1}^{n} \frac{x_i}{n\bar{x}} \log_b \frac{x_i}{n\bar{x}} \tag{3}$$

This yields $\mathbf{p} = (p_1, ..., p_n) = (\frac{x_1}{n\bar{x}}, ..., \frac{x_n}{n\bar{x}}) = \mathbf{r}/n$ on a simplex, however, **the interpretation of Theil's index is inconsistent.**

Theil's Version

Since

$$\frac{log_{b_0}\ t}{log_{b_1}\ t} = K,\ \forall t \tag{4}$$

say, K can serve as a conversion between information units b_0 and b_1. **This feature is elided from Theil's construction.**

Theil's Decomposition

Thiel's index, on a population of n total individuals, is commonly defined as:

$$T = \sum_{j=1}^{m} p_j \; R_j \; log_b \; R_j + \sum_{j=1}^{m} p_j \; R_j \; T_j \tag{5}$$

with m disjoint groups, $g_1, ..., g_m$, each with n_j members - $n = \sum_j n_j$., and

$$T_j = n_j^{-1} \sum_{i \in g_j} r_{ij} \; log_b \; r_{ij}, \tag{6}$$

Decomposition

Consider this rewrite of (5),

$$T = \sum_G \frac{n_g}{n} \frac{\overline{X}_g}{\overline{\overline{X}}} log_b \frac{\overline{X}_g}{\overline{\overline{X}}} + \sum_G \frac{\overline{X}_g}{\overline{\overline{X}}} \frac{1}{n_g} \sum_g \frac{X_{ig}}{\overline{X}_g} log_b \frac{X_{ig}}{\overline{X}_g} \tag{7}$$

Decomposition

The first term on the right hand side

$$Across = \sum_G \frac{n_j}{n} \frac{\overline{X}_g}{\overline{X}} \, log_b \frac{\overline{X}_g}{\overline{X}} \tag{8}$$

is the measure of the *across* or *between* group inequality; the second term

$$Within = \sum_G \frac{\overline{X}_g}{\overline{X}} \frac{1}{n_g} \sum_g \frac{X_{ig}}{\overline{X}_g} log_b \frac{X_{ig}}{\overline{X}_g} \tag{9}$$

the *within* group inequality.

Misspecification

Examine the expectation of the *within* term:

$$E[Within] = E[\sum_G \frac{1}{\overline{X}n_g} \sum_g E[X_{ig} log_b \frac{X_{ig}}{\overline{X}_g} | G = g]] \tag{10}$$

$$= E[\sum_G \frac{1}{\overline{X}n_g} E[\sum_g X_{ig} log_b \frac{X_{ig}}{\overline{X}_g}]] \tag{11}$$

$$\geq E[\sum_G \frac{1}{\overline{X}n_g} E[n_g\overline{X}_g log_b \frac{n_g\overline{X}_g}{n_g\overline{X}_g}]] \tag{12}$$

$$\geq E[\sum_G \frac{1}{\overline{X}n_g} E[0]] = 0 \tag{13}$$

Misspecification

An inspection of the across term,

$$E[Across] = \sum_G \frac{n_g}{n}\frac{n_g}{n_g}E[\frac{X_{ig}}{\overline{X}}log_b\frac{\overline{X}_g}{\overline{X}}] \tag{14}$$

$$= \sum_G \frac{n_g}{n}E[\frac{X_{ig}}{\overline{X}}log_b\frac{\overline{X}_g}{\overline{X}}] \tag{15}$$

$$= \sum_G \frac{n_g}{n\mu}E[X_{ig}log_b\frac{\overline{X}_g}{\overline{X}}] \leq \sum_G \frac{n_g}{n\mu}E[X_{ig}] \cdot E[log_b\frac{\overline{X}_g}{\overline{X}}] \tag{16}$$

Misspecification

Notice

- $\frac{\overline{X}_g}{\overline{X}} < 1$ implies
- $log_b \frac{\overline{X}_g}{\overline{X}} < 0$, and $\uparrow$ as $b \uparrow$
- And $\frac{\overline{X}_g}{\overline{X}} > 1$ implies
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Let $\epsilon_g^b \equiv E[log_b \frac{\overline{X}_g}{\overline{X}}]$

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Misspecification

In these situations

$$\sum_G \frac{n_g}{n\mu} E[X_{ig}] \cdot E[log_b \frac{\overline{X}_g}{\overline{X}}]$$

$$\begin{aligned} &= \frac{1}{n\mu}\{n_1 E[X_1] \cdot (-\epsilon_1^b) && (17)\\ &+ \cdots n_{m-1} E[X_{m-1}] \cdot (-\epsilon_{m-1}^b) && (18)\\ &+ n_m E[X_m] \cdot (\epsilon_m^b)\} && (19)\\ &\leq 0 && (20) \end{aligned}$$

Misspecification

- Reminder to draw a picture, if I can
- This happens in data where few, small groups (n_g) have high, or low, income relative to population size.
- Effect is inflated, non-linearly, by choice of b
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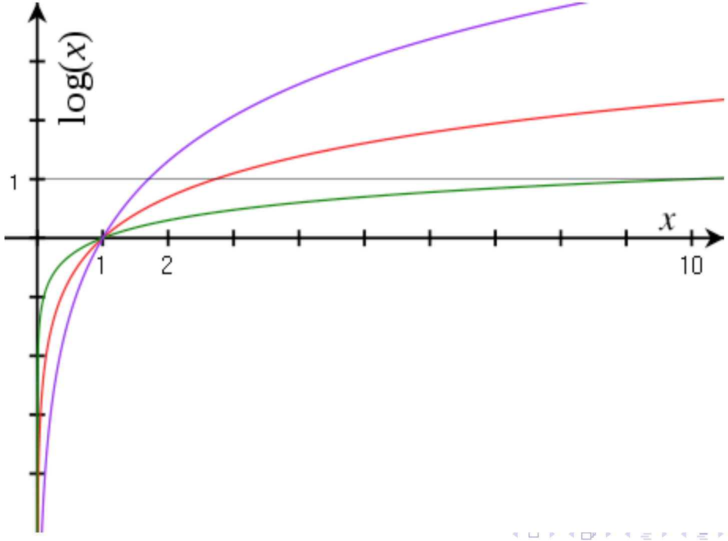

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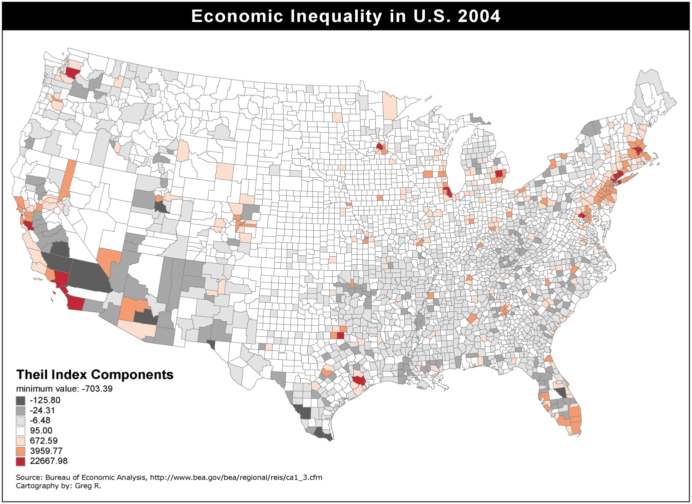

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- The probability of a particular event/realization is replaced with the income share for a particular individual.
- We were measuring prob mass of events, now we are measuring the 'size' of individual
- In this sense: The individual is the event, and the income share is the prob mass

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Comments

- The lack of a true event space (σ-algebra) yields a degenerate probability model...
- Though heuristic is consistent: *n*-dimensional simplex / Liouville family of distributions.
- But this belies a "deeper confusion" about Entropy and Probability [Jaynes (1965), *American Journal of Physics*].

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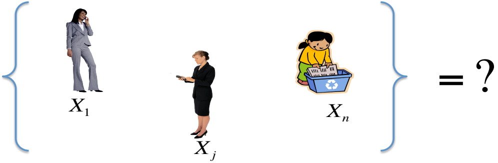
$$\Omega = \left\{ E_1 \quad E_j \quad E_n \right\}$$



Shannon's Axiomatization

Notice:

$$T = log(n) - H$$

and that Shannon's original proof specified

$$H = -K \sum_{i=1}^{n} p_i \, log \, p_i$$

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Skew-Sensitivity

One can arbitrarily choose partitions that guarantee across group sum dominates within groups sum.
Merely choose a collection G^* such that $x_j > n_j^{-1} \sum_{g_j} x_j$ for groups $g_1, ..., g_{[n/m]}$.
On this collection

$$A \equiv log \frac{n_j^{-1} \sum_{g_j} x_j}{n^{-1} \sum_G x_j} < log \frac{x_j}{n^{-1} \sum_G x_j} \equiv B$$

obscuring within group effects.

Log base characterizes 'Granularity'

As well, there exist collections G^* for every choice of b such that on G^* / g^*

$$A, B > 1$$

but

$$A, B < 1$$

on g^*

The Amnesia of (Probability) Measure

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Adjust b

In many settings — Rocke and Durbin Log-Linear hybridization; Box-Cox, Tukey families of transforms.. — b is a parameter to be estimated. $b = inf_g(R_g)$
Set

$$\hat{b} = min(r_g) = min_g(\frac{\overline{x}_g}{\overline{x}})$$

the minimum group income share.

Rank Transform

Essentially this is to revert Theil to Shannon...where $F_g = \mathbb{P}(group\ g)$ and $F_{X_g} = \mathbb{P}(X_g < x_g)$.
Set

$$Across \equiv \sum_g \hat{F}_g \cdot log_b(\hat{F}_g)$$

and

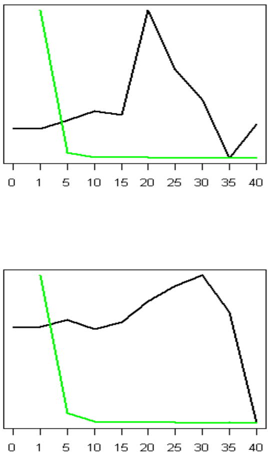
$$Within \equiv \sum_g \hat{F}_g \sum_{X_g} \hat{F}_{X_g} \cdot log(\hat{F}_{X_g})$$

Use $0 \leq \hat{F}_g, \hat{F}_{X_g} \leq 1$ or $\hat{F}_g, \hat{F}_{X_g} \in \{1, ..., m\}, \{1, ..., n_g\}$

Some Plots

Plots of Thiel index (black) and Across/Within ratio for normal random variates mean zero. Standard deviations: 0 to 40 - figure (a), 0 to 4.5 - figure (b), 0 to 4.5 figure (c). The index is generally increasing, though not monotonically, due to random draws - figure (a). Figure (b) illustrates tradeoff between across and within terms at small departures from uniform within group distribution. In figure (c) the log base $b = min(R_j)$, the minimum income share. The plots illustrate varied across/within ratio behavior, due to randomness; there is support for a non-decreasing across/within ratio.

Groups of Equal Mean Uniform Discrete Random Variables



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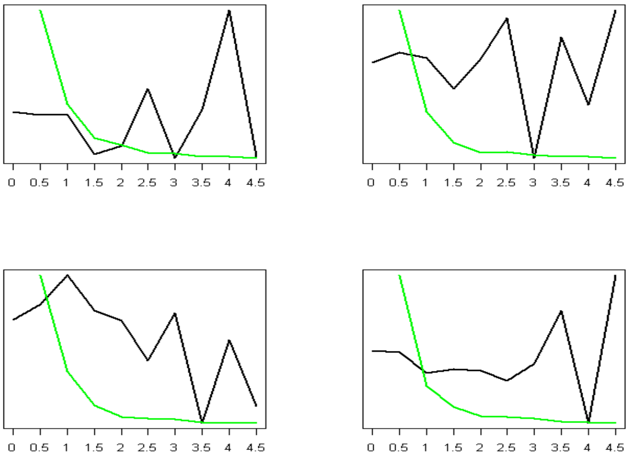
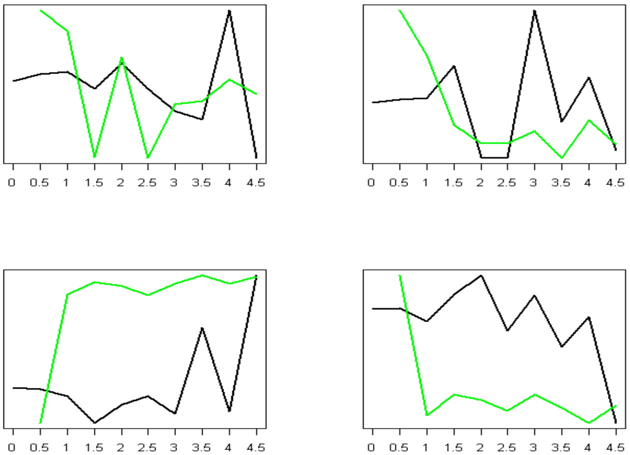


Groups of Equal Mean Uniform Discrete Random Variables, *b* is min.



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Equi-Inequalities

Several hypothesis of Equi-Inequality to consider:

I Each group is equally unequal
II Groups are equal
III Each group contributes equally to inequality measure
IV Inequality Across Groups is equivalent to Inequality Within Groups

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Hypothesis I, say

Take hypothesis I, for example, i.e. $T_g = T \; \forall g$

$$\frac{Across}{Within} \rightarrow T^{-1}\left(\frac{\sum_g n_g \overline{X}_g log_b(\overline{X}_g)}{\sum_g n_g \overline{X}_g} - log_b(\overline{X})\right)$$

Hypothesis I, say

Since T, $log_b(\overline{X})$ and are ancillary under I, the test statistic of interest is

$$\Phi_I \equiv \frac{\sum_g n_g \overline{X}_g log_b(\overline{X}_g)}{\sum_g n_g \overline{X}_g}$$

Hypothesis II, say

This is equivalent to the ordinary ANOVA or Bartlett's test for means or variances.

Hypothesis III, say

This is a generalization of I; the test statistic of interest is

$$\Phi_{III} = \sum_g \frac{n_g}{n} \frac{\overline{X}_g}{\overline{\overline{X}}} log_b(\frac{\overline{X}_g}{\overline{\overline{X}}})$$

Under III

$$\Phi_{III} \rightarrow m \cdot C$$

Hypothesis IV

This is the most general hypothesis; the test statistic is the ratio of across to within

$$\Phi_{IV} = \frac{\sum_G \frac{n_g}{n} \frac{\overline{X}_g}{\overline{\overline{X}}} \; log_b \frac{\overline{X}_g}{\overline{\overline{X}}}}{\sum_G \frac{\overline{X}_g}{\overline{\overline{X}}} \frac{1}{n_g} \sum_g \frac{X_{ig}}{\overline{X}_g} log_b \frac{X_{ig}}{\overline{X}_g}}$$

If b is well chosen, this a ratio of two positive quantities that is 1 at the null hypothesis. Similar to an F statistic.

Significance, Next

- Places Theil's contribution in statistical context, distributions on simplex (Dirichlet, Liouville)
- Codifies the index as a measure of distributional dependence
- Statistical properties/Hypothesis Tests are necessary descriptives for inequality, against inequality. 'Significance'
- Apply to HRS data: tests of change in inequality over time.

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