Statistical Issues for Re-Identification in Network Data

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Motivation

Repetitive Subscription Fraud

Telecommunications Industry

- Many people can't pay their bills, yet they still want telephone service.

Name	**Ted Hanley**
Address	**14 Pearl Dr** **St Peters, MN**
Balance	**$208.00**
Disconnected	**2/19/04 (nonpayment)**

Name	**Debra Handley**
Address	**14 Pearl Dr** **St Peters, MN**
Balance	**$142.00**
Connected	**2/22/04**

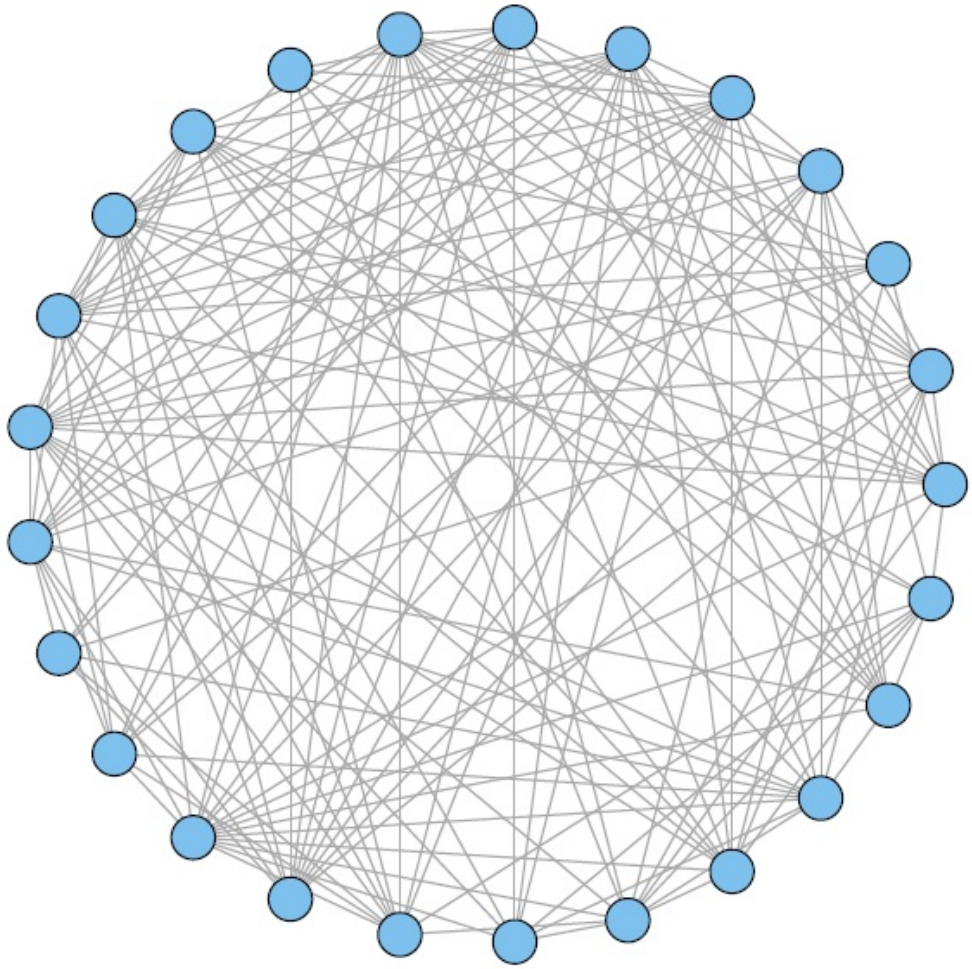
Name	**Elizabeth Harmon**
Address	**APT 1045** **4301 ST JOHN RD** **SCOTTSDALE, AZ**
Balance	**$149.00**
Disconnected	**2/19/04 (nonpayment)**

Name	**Elizabeth Harmon**
Address	**180 N 40TH PL** **APT 40** **PHOENIX, AZ**
Balance	**$72.00**
Connected	**1/31/04**

GOAL: Catch people who are taking part in fraudulent activities.

Motivation Continued

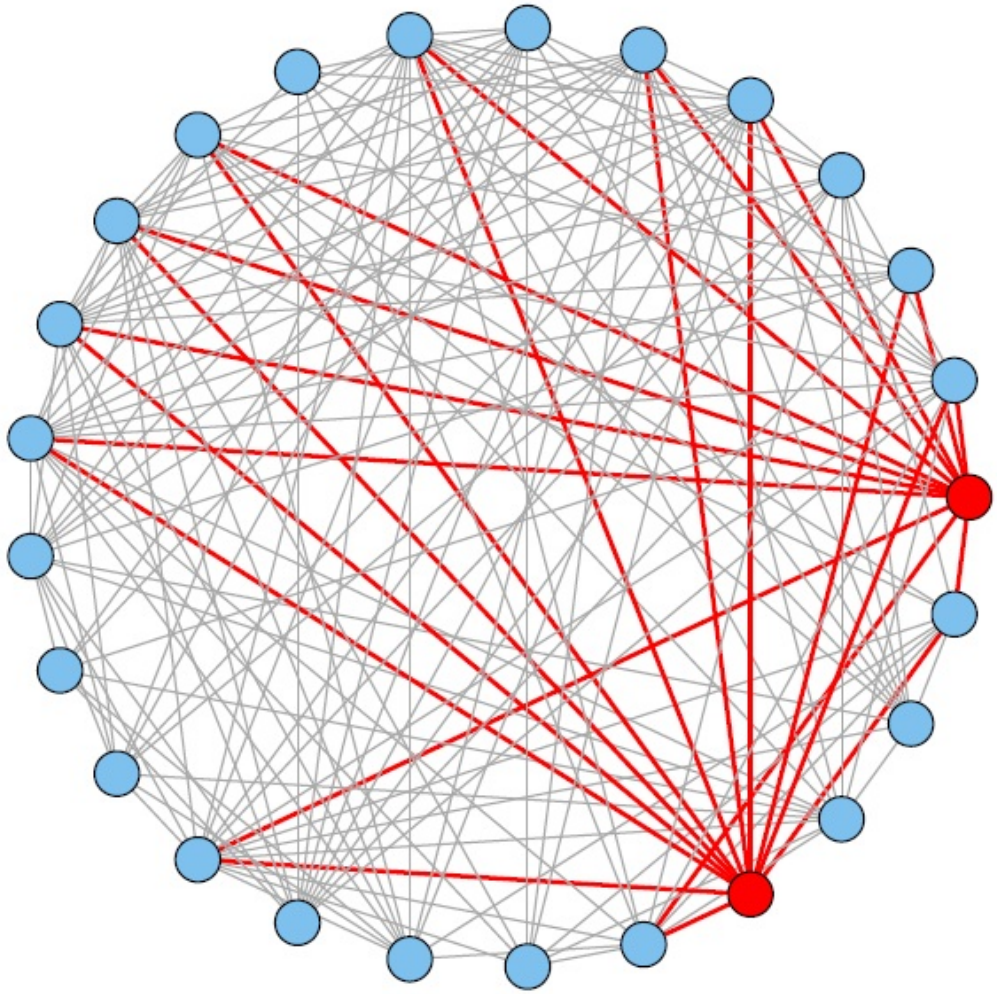
We observe a network with nodes representing phone users and edges representing a phone call between two uses.



Can you find the matching phone user?

Motivation Continued

The red nodes represent users who have the same call history.



Brief Background

Networks

- A *network* $G = (V, E)$ is a mathematical structure consisting of *nodes* V and *edges* E.
- Often easier to consider a network based on an *adjacency matrix*, $A = [a_{ij}]_{i,j=1}^{n}$, where $a_{ij} = 1$ if an edge exists between nodes i and j and $a_{ij} = 0$ if no such edge exists.
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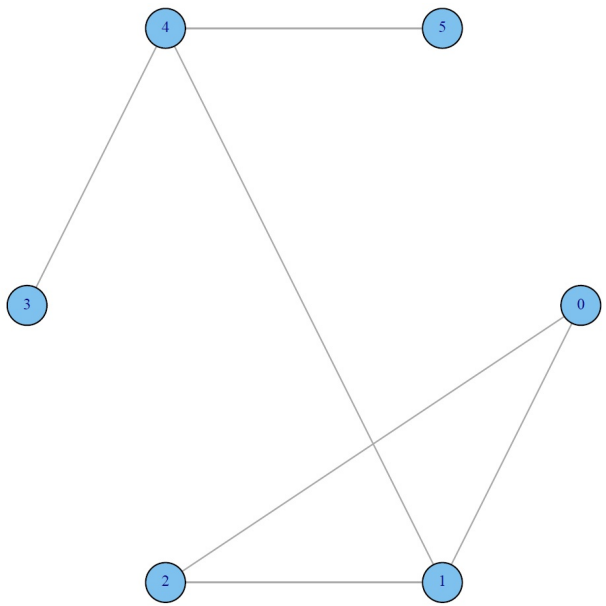

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$$d_4 = 3$$

Brief Background Continued

Erdös-Rényi Networks proposed in 1959

- Each edge is place independently with probability p
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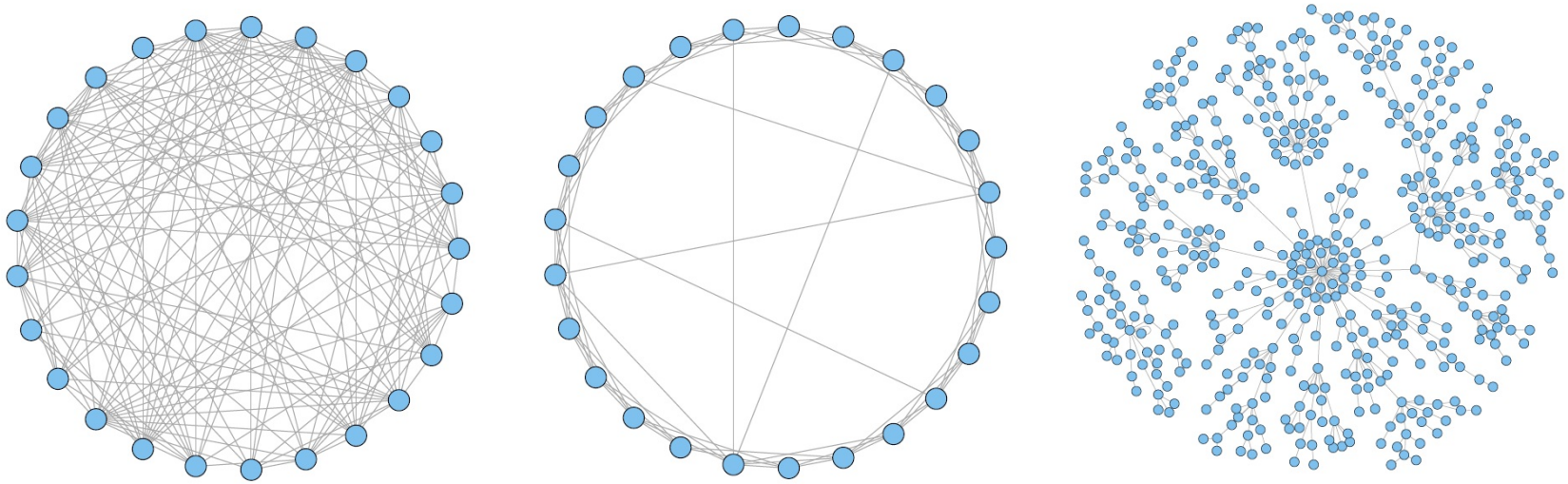
Barabási-Albert Scale-Free Networks proposed in 1999

- At each time step, a node enters the graph and connects to m nodes with probabilities proportional to node degree.
- The distribution of the degree decays slowly with $f_d \propto d^{-\alpha}$ for some constant α.
- Attribute scale-free phenomena to preferential attach and network growth

Brief Background Continued



Plots of Erdös-Rényi, Watts-Strogatz, and Barabási-Albert networks

Re-identification

- The *signature* of node i, denoted $\sigma(i)$, is the entity in which the node represents.
- As people interact with each other, they leave behind characteristic patterns of (a likely unique) behavior.
- The *re-identification problem* is the process of identifying two nodes have the same signature through this characteristic behavior.

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Previous Work

- C. Cortes , D. Pregibon, and C. Volinsky. Computational Methods for Dynamic Graphs. Journal of Computational and Graphical Statistics, 12:950–970, 2003.
- S. Hill , D. K. Agarwal , R. Bell, C. Volinsky. Building an effective representation for dynamic networks. Journal of Computational and Graphical Statistics, 15(3):584–608, 2006.
- S. Hill and A. Nagle. Social Network Signatures: A Random Graph Approximation for Re-Identification and Experimental Results. In Proceedings of Computational Aspects of Social Networks, pp. 23–33, 2009.

Methodology

- Let G be an observed network from a family of networks $\mathcal{G}_\theta$, e.g., Erdös-Rényi, small world, or scale free networks, where θ is a parameter that completely characterizes $\mathcal{G}_\theta$.

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- Let A be the corresponding adjacency matrix of G_θ.
- The *overlap score*, i.e., the number of neighbors that nodes i and j share is

$$S_\theta(i,j) = \langle \mathbf{a_i}, \mathbf{a_j} \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual dot product and $\mathbf{a_i}$ stands for the i^{th} row of A. When $i \neq j$, we call S_θ the *non-match score*.

Methodology Continued

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- For $i \neq j$, calculate $s(i,j)$. If this observed value is unusually large based on F_θ, we conclude that the *signatures* (identities) of i and j are the same.
- Formally, for all $i \neq j$, we perform the hypothesis test

$$H_0 : \sigma(i) \neq \sigma(j) \text{ vs. } H_1 : \sigma(i) = \sigma(j).$$

- We address the multiple hypothesis testing problem via controlling the false discovery rate (FDR).

Score Distributions: Erdös-Rényi Networks

To derive the *non-match score* distribution for nodes i and j,$i \neq j$, we consider the construction of the network.

Construction

1. Start with network consisting of n nodes and 0 edges.
2. For each pair of nodes (i, j), $i \neq j$, place an edge between them with probability p.
3. Once every pair of nodes is considered exactly once, the construction ends.

Score Distributions: Erdös-Rényi Networks Continued

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$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} s(1,2) &= \langle \mathbf{a_1}, \mathbf{a_2} \rangle \\ &= (0,1,1,0,0,0)(1,0,1,0,1,0)^T \\ &= 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 \\ &+ \quad 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \\ &= 1 \end{aligned}$$

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- The probability of success for each of these "trials" is

$$\mathbb{P}\{a_{i,i^*} \cdot a_{j,i^*} = 1\} = \mathbb{P}\{a_{i,i^*} = a_{j,i^*} = 1\} = p^2$$

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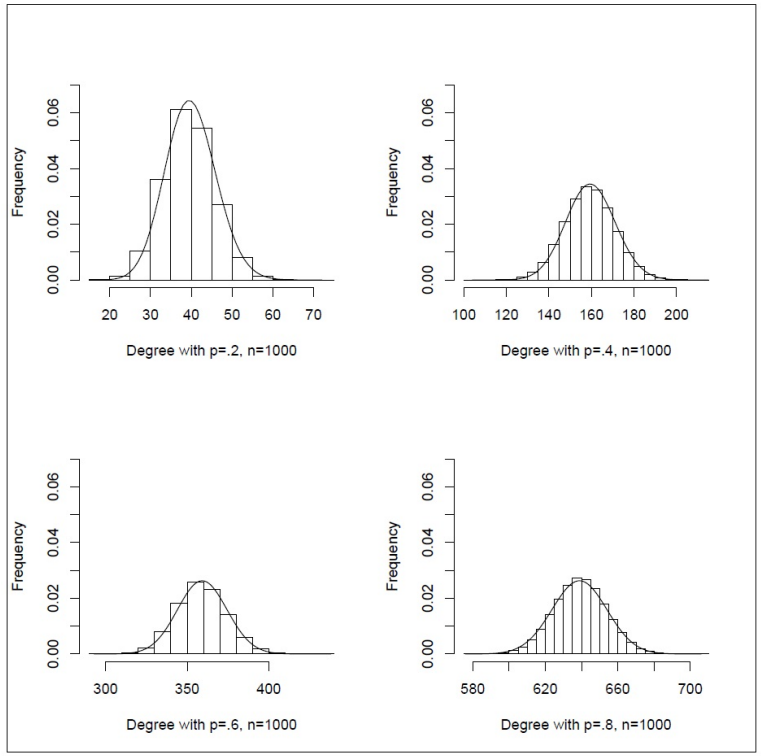
- The *non-match score* distribution is, therefore,

$$S_p(i,j) \sim \text{Bin}(n-2, p^2).$$

Score Distributions: Erdös-Rényi Networks Continued



Plots of non-match score distributions for Erdös-Rényi networks with varying p.

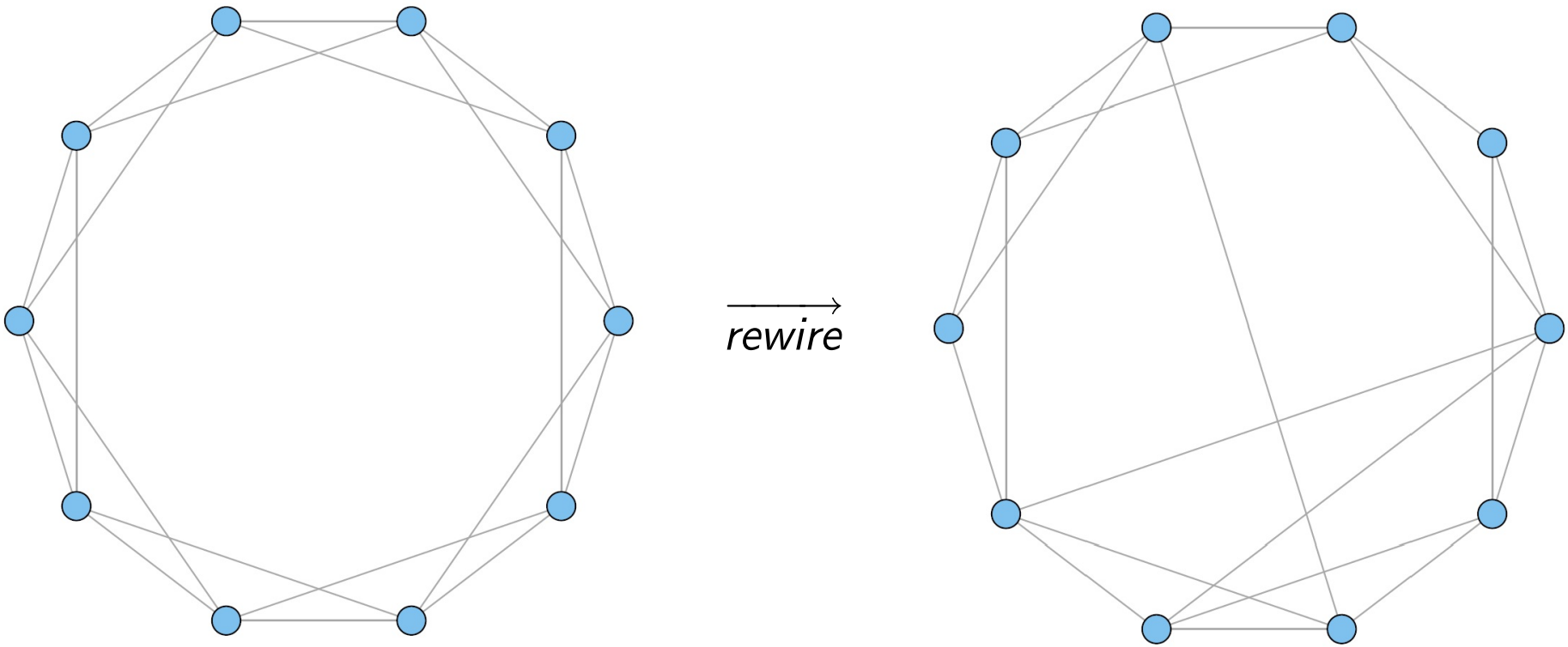
Score Distributions: Watts-Strogatz Small-world Networks

To derive the *non-match score* distribution for nodes i and j,$i \neq j$, we consider the construction of the network.

Construction

1. Start with circle of n nodes each connected to its k closest nodes on the left and right, giving a total of $2k$ edges for each node.
2. For each existing edge, decide to "rewire" it with probability p.
   - To rewire an edge, disconnect the edge from its right endpoint (node that lies clockwise). Then select one node based on a uniform distribution to reconnect to the right side of the edge to.
3. The algorithm ends when all original edges have been considered for rewiring exactly once.

Score Distributions: Watts-Strogatz Small-world Networks Continued



Score Distributions: Watts-Strogatz Small-world Networks Continued

To derive the *non-match score* distribution for nodes i and j,$i \neq j$, we need to consider three cases before rewiring.

1. nodes i and j are both unconnected to node i^*, i.e., $a_{i,i^*} = a_{j,i*} = 0$;
2. node i and j are both connected to node i^*, i.e., $a_{i,i^*} = a_{j,i^*} = 1$;
3. node i is connected to node i^* and node j is unconnected from i^*, i.e., $a_{i,i^*} = 1$, and $a_{j,i*} = 0$.

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For $i = 1$ and $j = 2$:

1. case 1 occurs 2 times ;
2. case 2 occurs 1 time;
3. case 3 occurs 3 times.

Score Distributions: Watts-Strogatz Small-world Networks Continued

The number of times each case occurs is:

$$n_1 = \begin{cases} n - 2k - ||i,j|| - 1, & \text{if } 1 \leq ||i,j|| \leq k \\ n - 2k - ||i,j|| + 1, & \text{if } k < ||i,j|| \leq 2k \\ n - 4k, & \text{if } ||i,j|| > 2k \\ 0, & \text{otherwise.} \end{cases}$$

$$n_2 = \begin{cases} 2k - ||i,j|| - 1, & \text{if } 1 \leq ||i,j|| \leq k \\ 2k - ||i,j|| + 1, & \text{if } k < ||i,j|| \leq 2k \\ 0, & \text{otherwise} \end{cases}$$

$$n_3 = \begin{cases} 2||i,j|| + 2, & \text{if } 1 \leq ||i,j|| \leq k \\ 2||i,j|| - 2, & \text{if } k < ||i,j|| \leq 2k \\ 4k, & \text{if } ||i,j|| > 2k \\ 0, & \text{otherwise.} \end{cases}$$

Note, each n_i depends on the distance between each pair of nodes before rewiring. The distance between nodes i and j is denoted $||i,j||$.

Score Distributions: Watts-Strogatz Small-world Networks Continued

- As before, each scalar product in the *score*, can be viewed as a Bernoulli trial were a success is defined by $a_{i,i^*} \cdot a_{j,i^*} = 1$, or $a_{i,i^*} = a_{j,i^*} = 1$
- We need to consider how a "success" arises for each case in order to derive the *non-match score* distribution.

Case 1

After the rewiring process, $a_{ii^*} = 1$ if

- one of the k edges considered for rewiring from node i is rewired to node i^*
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$$\mathbb{P}\{a_{i,i^*} = 1\} = \mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - (1 - p/n)^{2k}$$

$$f_1 \sim \text{Bin}(n_1, [1 - (1 - p/n)^{2k}]^2)$$

Case 2

After the rewiring process, $a_{ii^*} = 0$ if

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$$f_2 \sim \text{Bin}(n_2, [1 - p(\tfrac{n-1}{n})(1 - p/n)^{2k-1}]^2)$$

Case 3

After the rewiring process, $a_{ii^*} = 1$ if the above two cases hold appropriately. Therefore,

$$f_3 \sim \text{Bin}(n_3, [1 - p(\tfrac{n-1}{n})(1 - p/n)^{2k-1}][1 - (1 - p/n)^{2k}])$$

Score Distributions: Watts-Strogatz Small-world Networks Continued

The *non-match score* distribution is the convolution of random variables $X_1 \sim f_1, X_2 \sim f_2$, and $X_3 \sim f_3$,i.e.,

$$\begin{aligned}
\mathbb{P}\{S_\theta(i,j) = z\} &= \sum_{y=0}^{z} \binom{n_3}{z-y} p_3^{y-z}(1-p_3)^{n_3-(z-y)} \chi\{z-y \le n_3\} \\
&\times \sum_{x=0}^{y} \binom{n_1}{x} p_1^{x}(1-p_1)^{n_1-x} \chi\{x \le n_1\} \\
&\times \binom{n_2}{y-x} p_2^{y-x}(1-p_2)^{n_2-(y-x)} \chi\{y-x \le n_2\}
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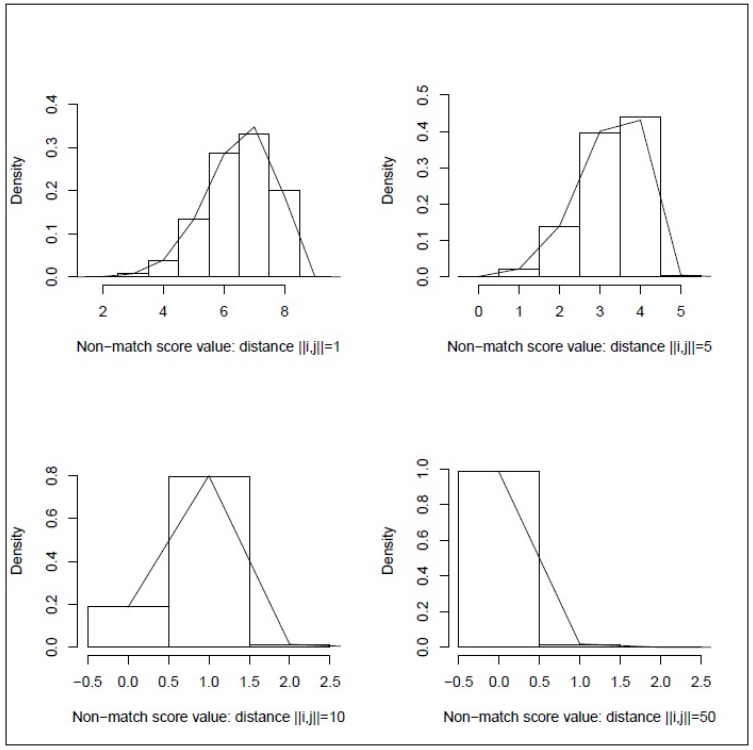
Score Distributions: Watts-Strogatz Networks Continued



Plots of non-match score distributions for Watts-Strogatz small-world networks

Score Distributions: Watts-Strogatz Networks Continued

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Score Distributions: Watts-Strogatz Networks Continued

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- To get around needing the labels, we view the data as coming from a mixture of all the *non-match score* distributions over all the possible distances between nodes.

Mixture Model

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- Let $\alpha_1, \ldots, \alpha_{\lfloor n/2 \rfloor} \in \mathbb{R}$ be the mixing parameters such that $0 \leq \alpha_i \leq 1$ for all i and $\sum_{i=1}^{\lfloor n/2 \rfloor} \alpha_i = 1$.

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$$f_{mix} = \sum_{i=1}^{\lfloor n/2 \rfloor} \alpha_i s_{||\cdot,\cdot||=i}.$$

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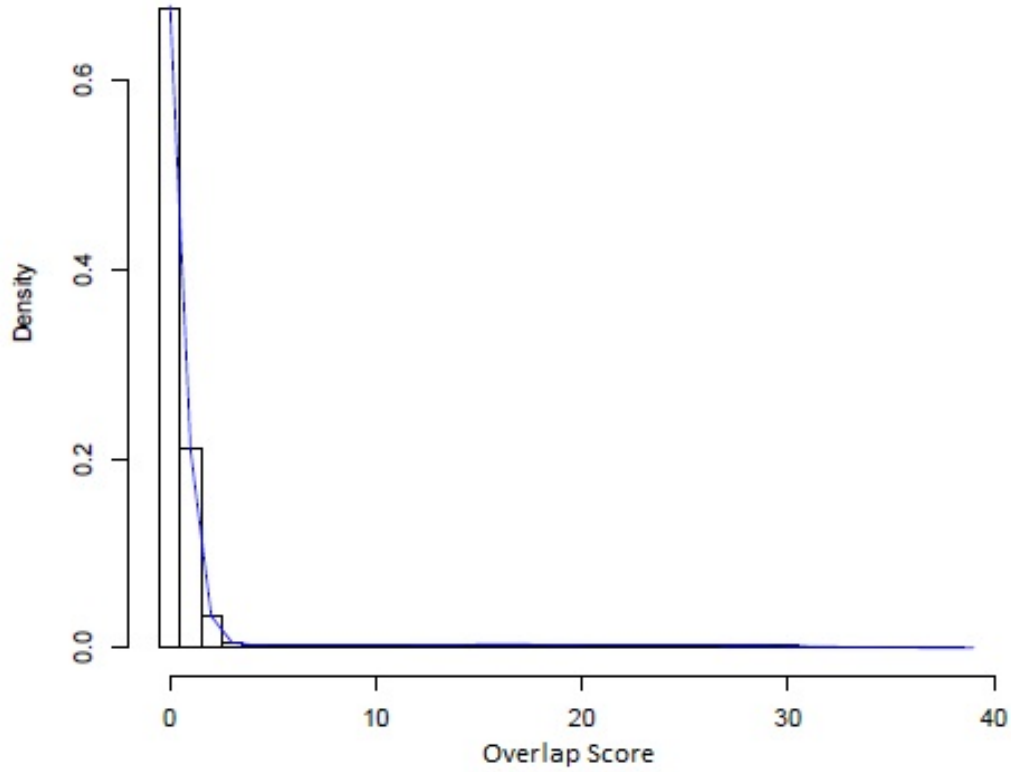
- The α_i's are actually known, so all we need to estimate is the parameters p and k as before.
  - For instance, if n is even, $\alpha_i = \frac{2}{n-1}$ for $i = 1, \ldots, \lfloor n/2 \rfloor - 1$ and $\alpha_{\lfloor n/2 \rfloor} = \frac{1}{n-1}$.

Score Distributions: Watts-Strogatz Networks Continued



Plots of non-match score mixture distribution for Watts-Strogatz small-world networks

Score Distributions: Barabási-Albert Scale-free Networks

Barabási and Albert provide a method of constructing scale-free networks based on growth and preferential attachment, however, their description is imprecise. Bollobás and Riordan remedy this issue by precisely specifying the model of Barabási and Albert.

Construction

1. Start with an initial graph with one vertex and one loop.
2. Let $d_{n,i}$ denote the degree of node i when the size of the graph is n. At each time step add node n together with a single edge between nodes n and i, where i is randomly chosen with

$$\mathbb{P}(i = s) = \begin{cases} d_{n-1,s}/(2n-1), & 1 \leq s \leq n-1 \\ 1/(2n-1), & s = n. \end{cases}$$

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This network can be generalized to add m edges with the entrance of each new node.

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The two key elements of this construction are growth and preferential attachment.

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- Growth: nodes are added to the network over time.
- Preferential attachment: the probability that a new edge is attached to a node is proportional to the nodes degree.

Score Distributions: Barabási-Albert Scale-free Networks Continued

Preferential attachment introduces a dependence in the network which is not present in the Erdös-Rényi or Watts-Strogatz networks. In particular,

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We consider the case when $m = 1$, making the third case irrelevant.

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Case 1

$$\begin{aligned}\mathbb{P}\{a_{i^*i}=1, a_{i^*j}=1\} &= \frac{4i^*+2}{(2j-1)(4(i^*)^2-1)}\prod_{s=i+1}^{j-1}\left(\frac{2s}{2s-1}\right) \\ &= \left(\frac{4i^*+2}{(2j-1)(4(i^*)^2-1)}\right)\left(\frac{4^{j-i-1}(j-1)!^2(2i)!}{(2j-2)!(i)!^2}\right)\end{aligned}$$

Case 2

$$\begin{aligned}\mathbb{P}\{a_{i^*i}=1, a_{i^*j}=1\} &= \frac{1}{(2i^*)(2j-1)}\prod_{s=i}^{j-1}\left(\frac{2s}{2s-1}\right) \\ &= \frac{1}{(2i^*)(2j-1)}\left(\frac{4^{j-i}(j-1)!^2(2i-2)!}{(2j-2)!(i-1)!^2}\right)\end{aligned}$$

Score Distributions: Barabási-Albert Scale-free Networks Continued

- Let $X_{i,j}^{i^*}$ denote the Bernoulli random variable with success probability

$$p_{i,j}^{i^*} := \mathbb{P}\left\{a_{i^*i} = 1, a_{i^*j} = 1\right\},$$

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- The non-match score distribution for specified nodes i and j is the convolution of random variables $X_{i,j}^{i^*}$ as i^* ranges from 1 to n,i.e.,

$$Z = \sum_{i^*=1}^{j-1} X_{i,j}^{i*}.$$

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Unlike the previous network constructions, these Bernoulli random variables are dependent.

The dependence in the Bernoulli random variables is that the sum is restricted to be one. Thus, Z is a Bernoulli random variable with probability of a success

$$\mathbb{P}\{Z=1\} = \mathbb{E}\left[\sum_{i^*=1}^{j-1} X_{i,j}^{i^*}\right] = \sum_{i^*=1}^{j-1} p_{i,j}^{i^*}$$

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The *non-match score* distribution for nodes i and j is

$$\mathbb{P}\{S_1(i,j)=s\} = \begin{cases} 1-\sum_{i^*=1}^{j-1} p_{i,j}^{i^*}, & \text{if } s=0 \\ \sum_{i^*=1}^{j-1} p_{i,j}^{i^*}, & \text{if } s=1. \end{cases}$$

Estimation

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- Many types of parameter estimation in networks are derived from the sampling scheme in which the data was collected.
- To get around this dependency and since we want the estimation to be valid no matter the sampling scheme, we use method of moments estimators.

Estimation: Erdös-Rényi Networks

- A family of Erdös-Rényi networks with known order n, along with the match and non-match scores distributions are completely characterized by parameter p.

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- Therefore, a moments estimator of p is

$$\hat{p} = \frac{2\mathcal{E}^{obs}}{n(n-1)},$$

where $\mathcal{E}^{obs}$ is the observed number of edges.

Estimation: Watts-Strogatz Small-world Networks

- For known n, the parameters p and k completely characterize this family of networks
- It was shown that $\mathbb{E}[\overline{d}] = 2k$, where $\overline{d}$ is the average degree. Thus,

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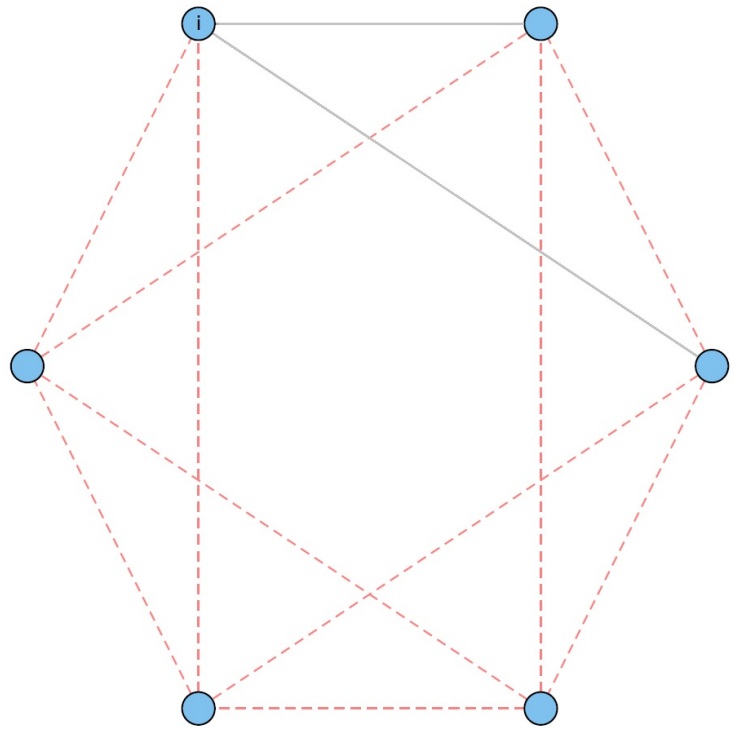
- Estimating p is more complicated.

Estimation: Watts-Strogatz Small-world Networks Continued

- Let t_i denote the total number of triads centered at node i
- Let t_i^{fixed} and t_i^{var} be the number of triads centered at i that always exist and the number of triads centered at i that vary based on the rewiring process, respectively.



Estimation: Watts-Strogatz Small-world Networks Continued

To calculate the expected value of t^{var} we consider the following.

- Let X_i be a random variable denoting the number of edges that are connected to node i initially that are not rewired to a different edge,

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- Let Y_i denote the number of edges that are not initially connected to node i that are rewired to node i,

$$\Rightarrow Y_i \sim \text{Bin}\left((n-2)k, p/n\right).$$

Estimation: Watts-Strogatz Small-world Networks Continued

$$
\begin{aligned}
\mathbb{E}[t_i] &= \mathbb{E}[t_i^{fixed}] + \mathbb{E}[t_i^{var}] \\
&= \mathbb{E}[t_i^{fixed}] + \sum_{a=1}^{k} \sum_{b=1}^{(n-2)k} \mathbb{E}[t_i^{var} | X_i = a, Y_i = b] \mathbb{P}[X_i = a, Y_i = b] \\
&= \mathbb{E}[t_i^{fixed}] + \sum_{a=1}^{k} \sum_{b=1}^{(n-2)k} \mathbb{E}[t_i^{var} | X_i = a, Y_i = b] \mathbb{P}[X_i = a] \mathbb{P}[Y_i = b] \\
&= \sum_{l=1}^{2k-1} l + \left(\sum_{a=1}^{k} \sum_{b=1}^{(n-2)k} (a+b)k + \sum_{c=1}^{a+b-1} c \right) \mathbb{P}[X_i = a] \mathbb{P}[Y_i = b] \\
&= \frac{k(k-1)}{2} + \left(\sum_{a=1}^{k} \sum_{b=1}^{(n-2)k} (a+b)k + \frac{(a+b)(a+b-1)}{2} \right) \\
&\times \mathbb{P}[X_i = a] \mathbb{P}[Y_i = b],
\end{aligned}
$$

Estimation: Watts-Strogatz Small-world Networks Continued

The total expected number of triad, $t_{tot} = \sum_{i=1}^{n} t_i = n \cdot t_i$, is

$$\begin{aligned} \mathbb{E}[t_{tot}] &= n\frac{k(k-1)}{2} + n\left(\sum_{a=1}^{k}\sum_{b=1}^{(n-2)k}(a+b)k + \frac{(a+b)(a+b-1)}{2}\right) \\ &\times \mathbb{P}[X_i = a]\mathbb{P}[Y_i = b]. \end{aligned}$$

Replace $\mathbb{E}[t_{tot}]$ and k with $\frac{\overline{d}}{2}$ with the observed number of total triads, and solve numerically for $\hat{p}$

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- The scale-free model considered here has only one parameter to be estimated, m.
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  1. taking m as the minimum degree;
  2. using the total number of edges to find m.
- This complication in parameter estimating m stems from the highly simplistic nature of the model.

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Theorem

The above procedure always controls the FDR at level less that or equal to $\frac{m_0}{m} q$, *where* m_0 *is the number of true null hypothesis.*

Simulation Results

Erdös-Rényi Networks

$n = 1000, p = .2$	FDR=0.00615; TPR=1
$n = 1000, p = .5$	FDR=0.00501; TPR=1
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Watts-Strogatz Small-World Networks

$n = 1000, k = 5, p = .001$	FDR=0.00100; TPR=0.99996
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Barabási-Albert Scale-free Networks

- For $m = 1$ this method has trouble identifying two nodes as the same.
- The problem arises since the degree of most of the nodes is just 1, which is not a rare value for the overlap score.

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  - Determine theoretical properties of the estimators or likelihood based methods for estimation and hypothesis testing.

Thank You

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Questions?