Joe, Harry. Multivariate Models and Dependence Concepts. (1997)

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MVEV applications working group

Outline

Summary

The main comments:

1. The extreme value limit of copulas is used to construct parametric families of extreme value copulas.
2. Extreme value copulas with generalized extreme value univariate margins are models for multivariate maxima.
3. Extreme value copulas with generalized extreme value univariate survival margins are models for multivariate minima.
4. Some (but not all!) families of extreme value copulas are obtained as the extreme value limits of other families.

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Total Positivity

Joe calls a nonnegative function b Totally Positive of order 2 if $b(x_1, x_2)b(y_1, y_2) \geq b(x_1, y_2)b(y_1, x_2)$, $x_i < y_i$. The multivariate extension for $\mathbf{X} \sim F$

$$F(\mathbf{x} \vee \mathbf{y})F(\mathbf{x} \wedge \mathbf{y}) \geq F(\mathbf{x})F(\mathbf{y})$$

Components matching (high-high,low-low).

Min/Max infinitely divisible

With $\mathbf{X} \sim F$, F^{γ} is generally a cdf for $\gamma \geq k-1$. If F^{γ} is a cdf for all $\gamma > 0$ then F is called Max-infinitely divisible.
To see: Say F is max-id then, $F^{1/n}$ is a cdf and if $(X_{i1}^{n}, ... X_{im}^{n}) \sim F^{1/n}$ iid then

$$\mathbf{X} \sim (max_i X_{i1}^{n}, ..., max_i X_{ik}^{n})$$

where the maxima are over the indices 1 to n. For min-id replace max by min and cdf F by survival function $\overline{F}$.

MEVD are limit distributions

Given $\mathbf{X} \sim F$
Take the component-wise maxima,

$$\begin{array}{ccccc} X_{11} & . & . & . & X_{1k} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ X_{n1} & . & . & . & X_{nk} \\ \hline M_{1n} & . & . & . & M_{kn} \end{array}$$

Multivariate Extreme Value Distributions (MEVD) come from the limits of the maxima: $M_{jn} = max_{1 \leq i \leq n} X_{ij}$.

via univariate GEVDs

Set $Z_i = \frac{M_{in} - a_{in}}{b_{in}}$ and then set
$G(\mathbf{z}) = \lim_n F^n(a_{1n} + b_{1n}z_1, ..., a_{kn} + b_{kn}z_k)$
to be the limiting distribution of the componentwise maxima, then

$$= \lim_n \mathbb{P}(M_{1n} \leq a_{1n} + b_{1n}z_1, ..., M_{kn} \leq a_{kn} + b_{kn}z_k)$$

can be written in terms of univariate GEV margins, via Sklar's theorem

$$= C(H_{\gamma_1}(z_1), ..., H_{\gamma_k}(z_k))$$

if H_γ is the ordinary univariate GEV approximation (a la 3-types theorem)

MEVD copula

As it turns out, with $u_j = H_\gamma(z_j)$ such that

$$C^t(\mathbf{u}) = C(u_1^t, ..., u_k^t)$$

if C is an MEV copula, and if we let $G = C(e^{-y_1}, ..., e^{-y_k})$ be a multivariate distribution with unit exponential (survival) margins

MEVD copula

$$C(e^{-ty_1},...,e^{-ty_k}) = C^t(e^{-y_1},...,e^{-y_k})$$

then with $A = -log(C(e^{-y_1},...,e^{-y_k}))$, G is a min/max stable multivariate exponential distribution, in that if $X_i \sim Exp$, then $\bigwedge_{i=1}^{k} \frac{X_i}{w_i} \sim Exp$.
As it turns out, min/max stable exponential distributions are MVE distributions.

Copulas on EVDs generate MEVDs

This yields

- A property of closure for min/max stable multivariate exponential distribution under weighted minima/maxima, analogous to the property of closure for under linear combinations for multivariate normal — with the $\wedge$ replacing the $+$ operator.
- A way of picking generator functions, $A = -logG$ where A is homogenous of order 1: $A(t\mathbf{x}) = tA(\mathbf{x})$.

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Copulas on EVDs generate MEVDs

- A way to generate MEVDs using univariate exponentials/GEVs [$Y = \frac{1}{X} \sim e^{-y^{-1}}$ is Frechet if $X \sim Exp$] (or survival functions) as argument since: min/max stable multivariate exponential distributions generate MEV copulas; MEVDs are min/max stable; the copula that results does not depend on univariate margins.
- MEVD copulas are easily recognized from the min/max stable representation being homogenous of order 1.

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Joe Copula

Starting with the Joe copula

$$F_\delta(x, y) = C(u, v) = 1 - ((1-u)^\delta + (1-v)^\delta - [(1-u)(1-v)]^\delta)^\delta$$

insert unit exponential unit survival margins. In the limit:

$$F^n(x + logn, y + logn) \sim [1 - n^{-1}(e^{-\delta x} + e^{-\delta y})^{1/\delta}]^n$$
$$\rightarrow e^{-(e^{-\delta x} + e^{-\delta y})^{1/\delta}}$$

Gumbel/Logistic Copula

Generates the Gumbel Copula (replace the exponential survival margins with general u, v or $1-u, 1-v$ (or (u,1-v), etc.):

$$C_\delta(u,v) = e^{-((-logu)^\delta+(-logv)^\delta)^{1/\delta}}$$

expressed in min-stable exponential form:

$$A_\delta(z_1, z_2) = (z_1^\delta + z_2^\delta)^{1/\delta}$$

Gumbel/Logistic Copula

The Gumbel copula is TP order 2 and in the so-called Archimedean family, meaning:

$$C_\delta(u, v) = \psi_\delta(\psi_\delta^{-1}(u) + \psi_\delta^{-1}(v))$$

with $\psi_\delta(s) = e^{-s^{1/\delta}}$.

Multivariate Generalization

This can be used for a straightforward extension to $k = 3$:

$$C(u_1, u_2, u_3) = \psi_{\delta_1}(\psi_{\delta_1}^{-1} \circ \psi_{\delta_2}(\psi_{\delta_2}^{-1}(u_1) + \psi_{\delta_2}^{-1}(u_2)) + \psi_{\delta_1}^{-1}(u_3))$$

with the constraint that $\delta_1 \leq \delta_2$

Multivariate Generalization

To extend to $k = m$ dimensions, apply the $k = 3$ generalization recursively, in terms of the min/max stable representation

$$A_{1,\ldots,m}(\mathbf{z}; \delta_1, \ldots, \delta_m) = ([A_{1,\ldots,m-1}(z_1, \ldots, z_{m-1}; \delta_1, \ldots, \delta_{m-1})]^{\delta_m} + z_m^{\delta_m})^{1/\delta_m}$$

with the constraint $\delta_1 \geq \cdots \geq \delta_m$.

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Comments

- Copulas which are easily generalizable may be restricted in dependence: TP_2, nested decreasing dependence in marginal sets.
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- MEVD copula may be inferior for dependence of MEVD in some settings (Smith comment re: Ledford, Tawn paper).

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